Particle trapping: A key requisite of structure formation and stability of Vlasov–Poisson plasmas

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Particle trapping: A key requisite of structure formation and stability of Vlasov–Poisson plasmas

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Particle trapping is shown to control the existence of undamped coherent structures in Vlasov–Poisson plasmas and thereby affects the onset of plasma instability beyond the realm of linear Landau theory. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4916774]

I. INTRODUCTION

In a recent work of Mouhot and Villani[1] (see also Ref. 2), the basic conditions have been explored under which Landau damping and its nonlinear analogon, the nonlinear Landau damping, take place in a collisionless Vlasov–Poisson plasma. In their perturbative analysis of both systems, based in lowest order on linear wave theory, a long-time mixing wipes out any structure such that the homogeneous unperturbed plasma state is approached time-asymptotically. It is hence obvious that deviations from these conditions are needed to allow structure formation. Whereas linear and nonlinear Landau damping scenarios require sufficiently quiescent plasmas and need analytic distribution functions for their perturbative description, coherent structures preferentially occur in driven, noisy plasmas, such as fusion plasmas, rely on a non-perturbative treatment, and require for their description non-analytic singular distribution functions.

On the other hand, there is a long tradition in plasma physics in its theoretical, experimental, and numerical treatment, which has dealt with various properties of nonlinear coherent structures as seen in a wealth of computer simulations and lab and space observations, respectively.

In the present investigation, we make use of this vast knowledge and deal with the opposite standpoint, namely, the question under which conditions long living inhomogeneous coherent structures can be established in a plasma and what the consequences for the plasma stability are. It can be understood as a supplement to earlier studies of phase space vortices or holes, such as Refs. 3–12, by exploring stronger singularities than those needed for the privileged class of cnoidal electron and ion holes, respectively.

The purpose of the paper is hence two-fold: (i) to enrich the class of cnoidal electron holes (CEHs) by allowing stronger singularities and (ii) to present a new view of the close relationship between trapping and coherency from which a new nonlinear route can be deduced of how a plasma can be destabilized well below linear threshold. One outcome will be that coherency and linearity exclude each other such that linear Landau theory cannot be used to describe a coherent, long-living wave pattern in phase space, no matter how small the wave intensity will be.

The paper, in its first part, is an investigation of to what extent the existence of undamped coherent structures is affected by physically suggested singularities, allowing a first glimpse on the function space needed for the description of structure formation processes in general.

In the second part, a new scenario of plasma destabilization is presented which becomes especially effective in the linearly subcritical regime. Relying on these equilibria, their non-perturbative description opposes the common picture of onset of instability as predicted by linear Landau theory.

II. DISTRIBUTION FUNCTION AND DENSITY

In the following, the matter of concern are 1D, coherent, stationary, electrostatic structures of weak amplitude of the form $\phi(x - v_0 t)$ where $\phi(x)$ and v_0 are the quantities to be determined. The distribution function $f(x - v_0 t, v)$ is prescribed and assumed to satisfy the Vlasov equation rather than the linearized Vlasov equation. All other aspects, such as the existence of trapped particles and the need to distinguish them from free particles, are then derived and established ones and not the result of further assumptions or approximations. There will be hence no way to omit this intimate correlation between "coherency" and "trapping," the main issue of the paper. In other words, coherency and stationarity in connection with the complete, untruncated evolution equation are strong demands on the state of a structurally excited plasma, strong enough to imply unambiguously trapping.[13]

Employing the pseudo-potential method,[3,14] our goal are hence stationary, 1D, electrostatic waves, which are travelling typically with a nonzero speed v_0 in a collisionless thermal plasma. In comparison with Ref. 15 and earlier papers cited therein, the present study is performed under generalized conditions as the focus here is on singularities of the distribution function of different kind, associated with trapping, and their impact on the existence of such solutions. The plasma we are dealing with consists of a simple two-component plasma with for convenience fixed ions and collisionless, mobile electrons which are subject to a resonant wave particle interaction. More complex plasmas with mobile ions or finite currents, for example, can be treated analogously, as shown previously in the already cited papers.

The electron motion in phase space is then governed by the Vlasov equation, which reads in the wave frame, i.e., in

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the frame moving with v_0 with respect to the laboratory frame,

$$[v\partial_x + \Phi'(x)\partial_v]f(x,v) = 0. \tag{1}$$

In (1), normalized quantities have been used, based on the density n_0 and the temperature T_e of the unperturbed plasma.

An appropriate, i.e., mathematically and physically meaningful, solution is given by the following Ansatz:[3–5,16]

$$f(x,v) = \frac{1+\frac{k_0^2\Psi}{2}}{\sqrt{2\pi}}\left[\theta(\epsilon)\exp\left[-\frac{1}{2}\left(\sigma\sqrt{2\epsilon}+v_0\right)^2\right] + \theta(-\epsilon)\kappa\exp\left(-\frac{v_0^2}{2}\right)\left(1+\gamma\sqrt{-\epsilon}-\beta\epsilon\right)\right]. \tag{2}$$

In (2), $\theta(z)$ represents the Heavyside step function, $\sigma = sg(v)$ is the sign of the velocity, and $\epsilon := \frac{v^2}{2} - \Phi(x)$ is the single particle energy, Φ the electrostatic potential. The first part in (2) represents the free (or untrapped), the second part the trapped electrons, separated by the separatrix in phase space, which is given by $\epsilon = 0$. The function $f(x, v)$, which depends on the two constants of motion, σ and ϵ (rather than on ϵ alone, as often found in the literature) is a solution of (1). In the limits of $\kappa = 1$, $\gamma = 0$, and of small amplitudes, it coincides with Schamel's distribution (2) (Ref. 15) under which the scenario of phase space hole and double layer solutions has been developed,[3–5,7] etc. It is hence an extension and permits a still broader range of nonlinear wave analyses being, as before, based primarily on physically stimulated distributions.

Notice that this distribution function experiences a jump across the separatrix when $\kappa \neq 1$. In case of $\kappa = 0$, trapped particles are completely absent and there is a void at the trapped particle region. The parameter κ is hence a measure for the strength of particle trapping involved during the formation process. Notice further that the absence of trapped particles ($\kappa = 0$) does not mean an absence of the trapping nonlinearity (TN) (see later).

To understand the specific form of (2), we note that the distribution function is suggested by the replacement of $v \equiv \sigma\sqrt{v^2}$ through $\sigma\sqrt{2\epsilon}$ and by the demand that it represents a shifted Maxwellian in the unperturbed state. In case of a perturbation, this property is transformed to the point where the trapped particles are absent, i.e., where $\Phi = 0$. We have thereby assumed without loss of generality that Φ satisfies $0 \leq \Phi(x) \leq \Psi$, where Ψ represents the amplitude of the perturbation. With this, we have correctly incorporated the unperturbed plasma state given by $\Psi \equiv 0$, being represented by the shifted Maxwellian: $f_M(v) = \frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}(v+v_0)^2]$. The propagating structures, we are looking for, are, therefore, embedded in an unperturbed thermal plasma. Other realistic background distributions, as, e.g., superthermal distributions,[17,18] are feasible but are not followed further here to keep the analysis as simple as possible.

In case of a continuous distribution ($\kappa = 1$) and a regular trapped particle distribution ($\gamma = 0$), the parameter β in (2) allows a fine tuning of the trapped particle state. It turns out, as shown earlier, e.g., in Ref. 15 and the papers cited therein, to be a necessary requisite for obtaining closed self-consistent descriptions. A dip in the distribution function in the trapped particle region in phase space, $\epsilon < 0$, is thereby provided by β negative.

The distribution (2) for $\kappa = 1$ and $\gamma = 0$, is nevertheless, singular at the separatrix, $\epsilon = \frac{v^2}{2} - \Phi(x) = 0^+$, i.e., when one approaches from the free particle side, but it is regular when one comes from the trapped particle side. A singularity occurs in the first derivative in v, which behaves like $1/\sqrt{\epsilon}$ as ϵ tends to 0^+, being hence a mild and integrable but unavoidable singularity. Note that the singularity in $\partial_v f(x,v)$, introduced by the free particle distribution, is characteristic for propagating structures with $v_0 \neq 0$ only and disappears for standing structures, where $v_0 = 0$. In (2), we have incorporated singularities in the trapped particle region not stronger than that introduced by the free particle region, and refer to still stronger singularities, which have been discussed in the literature, at the end.

The stronger singularities, i.e., when $\kappa \neq 1$ and $\gamma \neq 0$, are the focus of our attention in this letter.

To close the system, i.e., to find a self-consistent solution, we have to solve the second part, the Poisson equation, which in the immobile ion limit becomes

$$\Phi''(x) = \int f(x,v)dv - 1 =: -\mathcal{V}'(\Phi). \tag{3}$$

In the second step of (3), we have introduced the pseudo-potential $\mathcal{V}(\Phi)$, because its knowledge allows by a quadrature to obtain the final shape of the potential structure $\Phi(x)$ via the pseudo-energy

$$\frac{1}{2}\Phi'(x)^2 + \mathcal{V}(\Phi) = 0, \tag{4}$$

where we without loss of generality have assumed $\mathcal{V}(0) = 0$.

The electron density in (3), valid for small amplitudes $\Psi \ll 1$, can be obtained by a Taylor expansion of (2), followed by the velocity integration, as was done in Refs. 4, 8, and 10. It becomes

$$n(\Phi) = \left[1 - \frac{2}{\sqrt{\pi}}(1-\kappa)e^{-v_0^2/2}\sqrt{\Phi} + \frac{k_0^2\Psi}{2} + A\Phi - \frac{4}{3}b(v_0,\beta\kappa)\Phi^{3/2} + ...\right]. \tag{5}$$

where

$$A = \left[\frac{\kappa\gamma\sqrt{\pi}}{2}e^{-v_0^2/2} - \frac{1}{2}Z_r'\left(v_0/\sqrt{2}\right)\right], \tag{6}$$

$$b(v_0,\beta) = \frac{1}{\sqrt{\pi}}\left(1-\beta-v_0^2\right)e^{-v_0^2/2}. \tag{7}$$

A remarkable property of (5) is that it represents a Taylor expansion in powers of $\sqrt{\Phi}$, which is a consequence of trapping and of the TN. The case of $\kappa = 1$ and $\gamma = 0$, when the nonlinearity is represented by the $\Phi^{3/2}$ term, was the subject of our previous studies (see Ref. 15 and references cited therein), as mentioned. Here, we concentrate on the effect of the stronger nonlinearity $\sim\sqrt{\Phi}$ stemming from the discontinuity of $f(x, v)$ at the separatrix and on the singularity due to a non-analytic trapped particle distribution, when $\gamma \neq 0$.

Note that in (5), the higher order square nonlinearity $\sim\Phi^2$ has already been neglected. It would have been the ruling nonlinearity in an ordinary perturbation analysis that is free of trapping effects. In this density expression, the term $-\frac{1}{2}Z_r'(v_0/\sqrt{2})$ can be interpreted as an electronic shielding term[7] and is defined by $-\frac{1}{2}Z_r'(v_0/\sqrt{2}) := P\int \frac{1}{v}\partial_v f_M(v)dv$, where P stands for principal value, and $Z_r(x)$ represents the real part of the complex plasma dispersion function for real arguments. A plot of the function $-\frac{1}{2}Z_r'(x)$, displayed in Fig. 1, shows that it has a zero transition at $x_0 = 0.924$ ($\sqrt{2}x_0 = 1.307$), a minimum of -0.285 at $x_{min} = 1.506$ ($\sqrt{2}x_{min} = 2.13$), and is positive for $x < x_0$ and negative for $x > x_0$ and vanishes at infinity. Since, according to (6), it holds $x = v_0/\sqrt{2}$, there hence exist two separated regions for the phase velocity: a slow one with $0 \le v_{0s} \le 2.13$ and a fast one with $2.13 \le v_{0f}$.

III. NONLINEAR DISPERSION RELATION (NDR) AND PSEUDO-POTENTIAL

A self-consistent solution of our problem is then obtained by demanding

(i) $\mathcal{V}(\Phi) < 0$ in $0 < \Phi < \Psi$ and
(ii) $\mathcal{V}(\Psi) = 0$,

the latter condition representing zero electric field at potential maximum. After substitution of (5) into (3) and a subsequent Φ-integration, we get for the pseudo-potential $\mathcal{V}(\Phi)$

$$-\mathcal{V}(\Phi) = -\frac{4(1-\kappa)}{3\sqrt{\pi}}e^{-v_0^2/2}\Phi^{3/2} + \frac{k_0^2\Psi}{2}\Phi + \frac{A}{2}\Phi^2 - \frac{8}{15}b(v_0, \beta\kappa)\Phi^{5/2} + \dots \quad (8)$$

Condition (ii) then becomes

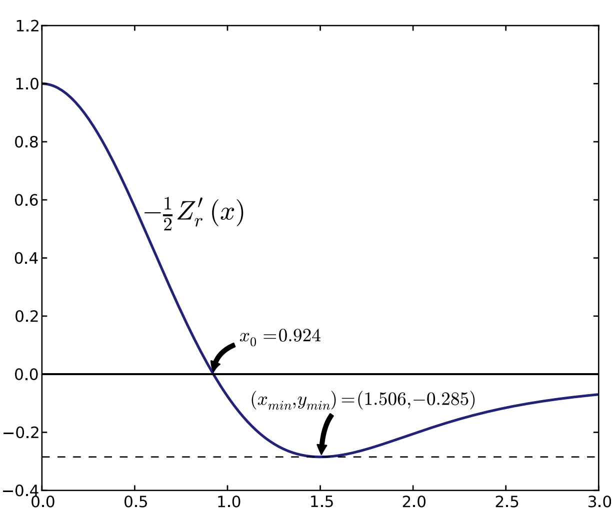

FIG. 1. The function $-\frac{1}{2}Z_r'(x)$.

$$\frac{8}{3\sqrt{\pi\Psi}}(1-\kappa)e^{-v_0^2/2} = k_0^2 + A - \frac{16}{15}b(v_0, \beta\kappa)\sqrt{\Psi} + \dots \quad (9)$$

This is the NDR as it is the determining equation for the phase velocity v_0 in terms of the other parameters.

By substitution of A from (9) into (8) the pseudo-potential can be rewritten as

$$-\mathcal{V}(\Phi) = -\frac{4(1-\kappa)}{3\sqrt{\pi\Psi}}e^{-v_0^2/2}\Phi^{3/2}\left(\sqrt{\Psi}-\sqrt{\Phi}\right) + \frac{k_0^2}{2}\Phi(\Psi-\Phi) + \frac{8}{15}b(v_0, \beta\kappa)\Phi^2\left(\sqrt{\Psi}-\sqrt{\Phi}\right) + \dots \quad (10)$$

It is interesting to note that $\mathcal{V}(\Phi)$, which is controlled by κ, k_0, and β but not by γ, is composed of three individual constituents each of which representing substructures in appropriate limits. It may serve as the building block in studies of phase-space turbulence in which an ensemble of kinetic structures replaces an ensemble of linear waves.[19–22] The last term in (10) represents a $\sec h^4(x)$ solitary wave, the second term a harmonic $(1+\cos(x))/2$ wave, and the first term a periodic $\cos^4(x)$ solution, when $\kappa > 1$ (see later). Note that γ enters through A in the NDR (9) only.

From (9) and (10), it is easily seen that in the limits of $\kappa = 1$ and $\gamma = 0$, when the distribution function is continuous and the singularity is missing in the trapped particle distribution, we end up in (7) and (8) of Ref. 15. Our generalized ansatz hence includes the nonlinear cnoidal electron hole modes, described in Ref. 15 and the earlier papers cited therein, as the smoothest version.

Since our focus in this letter is the impact of distributions with a stronger singularity, resulting from trapping, we concentrate on cases where the last term in (9) and in (10) can be neglected.

IV. VARIOUS WAVE SOLUTIONS

First we note that the parameter k_0 accounts for periodic wave solutions and that localized solitary wave solutions at least require $k_0 = 0$. This is easily seen from (5) in the minimum point $\Phi = 0$, where it holds $\Phi'' = \frac{k_0^2\Psi}{2}$. A nonzero k_0 hence gives rise to a positive curvature of Φ at its minimum and hence to a periodic wave solution (We mention in parenthesis that the wavelet solution, uncovered in Ref. 15, belongs to this latter category. It represents another type of localized wave structure, being ubiquitously met in space plasmas.). Note also that $k_0 = 0$ can still represent a periodic solution, namely, when the structure is concentrated around potential maximum with vanishing curvature at potential minimum, such as for $\Phi(x) \sim \cos^4(x)$ (see Appendix A).

A. Discontinuous distribution function

The simplest case, therefore, corresponds to a solitary wave solution ($k_0 = 0$) and to a complete absence of trapped electrons ($\kappa = 0$), which of course does not imply the absence of TN. For this to happen, we should have solved the NDR

$$\frac{8}{3\sqrt{\pi\Psi}}e^{-v_0^2/2} = -\frac{1}{2}Z_r'\left(v_0/\sqrt{2}\right), \tag{11}$$

with

$$-\mathcal{V}(\Phi) = -\frac{4}{3\sqrt{\pi\Psi}}e^{-v_0^2/2}\Phi^{3/2}\left(\sqrt{\Psi}-\sqrt{\Phi}\right). \tag{12}$$

This $\mathcal{V}(\Phi)$ is, however, strictly positive in $0<\Phi<\Psi$ and hence disqualified as a possible pseudo-potential. A solitary wave solution can, therefore, not exist without trapped electrons.

Extending next the search to periodic wave solutions, we have to solve

$$\frac{8}{3\sqrt{\pi\Psi}}e^{-v_0^2/2} = k_0^2 - \frac{1}{2}Z_r'\left(v_0/\sqrt{2}\right) > 0, \tag{13}$$

with

$$-\mathcal{V}(\Phi) = -\frac{4}{3\sqrt{\pi\Psi}}e^{-v_0^2/2}\Phi^{3/2}\left(\sqrt{\Psi}-\sqrt{\Phi}\right) + \frac{k_0^2}{2}\Phi(\Psi-\Phi). \tag{14}$$

The positive new term in (14) raises the chance of getting a solution.

To see as to whether a solution is possible, we consider $\mathcal{V}(\Phi)$ at the right, most critical border. It holds for $\Phi \to \Psi^-$

$$-\mathcal{V}(\Phi) = \left(\sqrt{\Psi}-\sqrt{\Phi}\right)\Psi^{3/2}\left[k_0^2 + \frac{1}{2}Z_r'\left(v_0/\sqrt{2}\right)\right]\Big/2,$$

where use has been made of (13). This expression as well as both sides of (13) should be positive which can be satisfied when k_0^2 is sufficiently large, say, of order unity. The smallness of Ψ in (13) is then guaranteed by sufficiently large phase velocities.

We conclude that in the complete absence of particles in the trapped range (a void in phase space), only periodic structures with sufficiently high phase velocities and short wavelengths are admitted. Solitary waves cannot exist under zero trapping conditions.

This property can be transmitted to a partial filling of the trapped range when κ is nonzero but less than unity, since in view of (9), (10) (and in the absence of the $b(v_0, \beta\kappa)$ terms) the same situation prevails. The possibility of an overpopulation of trapped particles for $k_0 = 0$ is treated in the Appendix, where it is shown that strictly speaking, only a periodic structure can exist, not a pure solitary one.

Our conclusion, therefore, is that discontinuous distributions exclude solitary wave structures.

B. Trapped particle singularity

Coming now to a softer singularity at the separatrix by demanding a continuous distribution function, $\kappa=1$, we see from (10) that only a harmonic wave, represented by the second term on the right hand side, does survive. So, the last step is to learn how the phase velocity of this sinusoidal wave is affected by a singular trapped particle distribution with $\gamma \neq 0$.

The NDR (9) becomes $k_0^2 + A = 0$ with A given by (6) and reads

$$k_0^2 - \frac{1}{2}Z_r'\left(v_0/\sqrt{2}\right) = B, \tag{15}$$

with $B = -\frac{\gamma\sqrt{\pi}}{2}e^{-v_0^2/2}$. It is hence of the same form than the NDR (7) of Ref. 15 with a different B. When $B=0$, or $\gamma=0$, respectively, it gives rise to a hook-like dispersion relation, the well-known "thumb-curve,"[4] with two branches, the fast Langmuir branch and the slow electron acoustic branch. It is purely nonlinear, resembles the van Kampen dispersion relation with $\lambda=0$ (Refs. 10 and 15) and is also called "on-dispersion curve." When $B \neq 0$, or $\gamma \neq 0$, we get, depending on its value, a multitude of dispersion curves, also called "off-dispersion curves." Fig. 2 shows both types of curves.

Note that for $B>0$ and the fast branch, there is a cut-off at lower k_0 given by $\sqrt{B} < k_0$. The dashed line is the separating line between the fast and slow branches. For $B<0$, no solution exists for $k_0 < \sqrt{\frac{-B}{2}}$. The minimum possible B is given by $B_{min} = -0.19$, which is located at $\sqrt{2}k_0 = 0.436$.

For negative γs another restriction comes from the nonnegativity of the trapped particle distribution, which reads $1+\gamma\sqrt{\Psi} > 0$.

Although the NDR here is identical with the one in Ref. 15, there is a big distinction in the shape of the associated electric potential. Whereas the potential here is a single harmonic for any B, it is multi-harmonic for $B \neq 0$ in the other case, when B is related with $b(v_0, \beta)$.

The smoother behavior of the distribution at the separatrix for the latter case, the cnoidal electron hole, has the unambiguous consequence of a high harmonic content with a multitude of phase-locked modes.[10]

This implies that a potential, the Fourier decomposition of which showing a large spectrum of modes, must have been created by a smoother electron distribution, for which not only $\kappa=0$ but also $\gamma=0$. With a macroscopic measurement of $\Phi(x)$ one can hence shed light on the microscopic plasma state.

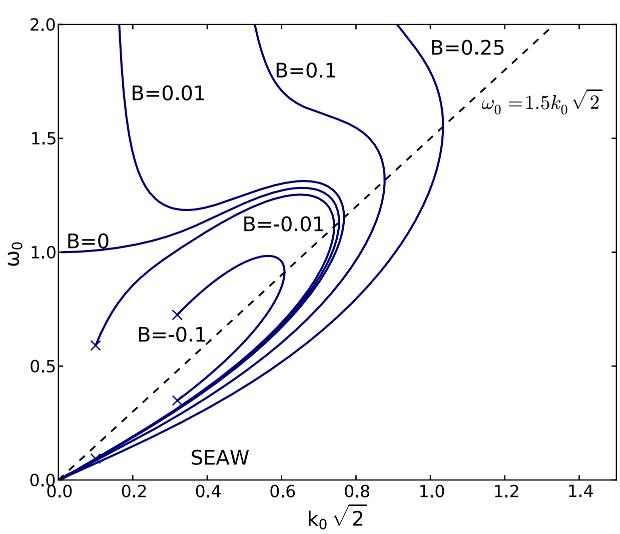

FIG. 2. NDR with $\omega_0 := k_0 v_0$.

Another question we like to address is where a typical BGK mode fits in. This of course depends on how much of information about $\Phi(x)$ can be invested. As the BGK method[23] is as general as the pseudo-potential method, a complete knowledge of the spectral content and phase velocity of $\Phi(x - v_0 t)$ of a cnoidal hole solution would result in a trapped particle distribution, where only the β term in (2) contributes. But, this *a priori* information is not available without a preceding pseudo-potential analysis of the present type (see also Ref. 15). A consequence is that one has to cope within the BGK method with simpler ansätze for $\Phi(x - v_0 t)$ at the cost of the regularity of $f_{et}(x - v_0 t, v)$.

A typical example treated by several authors[24–27] are solitary potentials of Gaussian- or sec h^ν-type, $\nu > 1$, which give rise to a $\sqrt{-\epsilon}\ln(-\epsilon)$ singularity of the trapped particle distribution, being hence even more singular than the ones treated here.

As a rule, the simpler the potential is chosen in terms of the harmonic spectrum, the more singular the distribution will come out.

Since, however, singularities are non-physical, being wiped out by additional processes such as phase space diffusion, coarse graining, and discretization in numerics, it is expected that the cnoidal hole solutions, exhibiting the smoothest distributions, will be the ones coming closest to the final description and are hence privileged.

Before we address the fundamental relationship between Vlasov equilibria and Vlasov stability, let us summarize:

In the first part of the paper, we have concentrated on trapped electron equilibria being associated with rather strong singularities stemming from the trapped electron distribution function. They have to be added to the smoother and well documented class of CEHs with the solitary electron hole[5] as the most prevalent member. Ions have been treated immobile, so far. The zoo of structures is, however, much wider encompassing not only structures involving ion trapping, namely, in cases where finite ion mass is taken into account and where the phase velocity is close to ion thermal velocity, but it includes also structures which at first glance seem to exist without the need for a kinetic treatment and trapping. An example is the ion acoustic soliton.[28] In Appendix B, we will show how trapping is coming in through the back door such that this mode has to be incorporated in this zoo, as well. More generally, it would be of interest to learn to what extent linear electrostatic structures in a thermal, magnetized plasma, such as kinetic shear Alfvén wave structures,[29] which in their dispersion relation exhibit a $Z'(\omega/k)$ term, representing Landau resonance, face the same problem as our present Vlasov–Poisson modes, as a result of coherency, and should, as undamped modes, be handled in the same way namely, nonlinearly by the inclusion of trapping effects, as will be suggested also from Sec. V. Another example may be the mutual interaction of drift waves and zonal flows in drift wave turbulence,[30] where the trapping nonlinearity in connection with coherency and stationarity can enter for the electrons as an extension to the Boltzmann response in their parallel dynamics and for the ions by a replacement of the linear ion Landau damping term through the trapped ion nonlinear term. And also the trapping of fluid elements in the $E \times B$ shear flow dynamics described by a convective cell equation (or more sophisticated ones) for the vorticity field provides a further example.[11,15] Such issues are delegated to forthcoming investigations.

V. SUBCRITICAL PLASMA DESTABILIZATION BY HOLE EQUILIBRIA—THE FAILURE OF LANDAU APPROACH

What does this extensive class of trapped particle equilibria (TPE) mean in the context of plasma theory? No more and no less than that Landau theory gets a rival with respect to the onset of plasma instability, as will be explained now.

First, each member of TPE is by definition lying on the border in function space separating damped from growing wave solutions, very much similar to a van Kampen mode, which separates such solutions in the framework of linear Vlasov theory. Indeed, Landau in his famous 1946 paper (Ref. 31), in which he applied Fourier–Laplace technique to derive time-asymptotically a dispersion relation, made use of continuity at $\gamma_L = 0$, where γ_L is the linear growth rate. He established, by invoking continuity, a single expression for γ_L, namely, $\gamma_L = \frac{\pi\omega_p^3}{2k^3 n_0} f_0'\left(\frac{\omega_r}{\sqrt{2}kv_{th}}\right)$, valid for both, damped, and growing solutions, and thereby defined the Landau contour. His instruction, where $\gamma_L = 0$ is related to a van Kampen mode, is undoubtedly legitimate within the context of linear Vlasov theory. It fails, however, when seen from the full untruncated Vlasov equation. The reason is that a single monochromatic and hence coherent van Kampen mode satisfies only the linear but not the full nonlinear Vlasov equation.[10,15] (To be a solution of both, $\partial_v f_1(x, v, t)$ has to vanish at resonant velocity, but it does not. Another shortcoming of a van Kampen mode (and a Landau mode) is that f_1 is strongly singular due to the Cauchy principle value and the delta function singularity, violating the necessary linearization condition $|\partial_v f_1| \ll |\partial_v f_0|$ in the resonant region and hence introducing an inconsistency to the linearized treatment of the Vlasov equation, see also Sec. III of Ref. 15.)

To be valid, Landau theory has to rest on an incoherent superposition of many van Kampen modes with different k and different phases, predominantly random phases, such as in quasilinear theory. Only when a single harmonic mode is embedded in a broad-band spectrum of incoherent waves, then coherency (and hence trapping) can be suppressed. Phase mixing rather than phase locking then rules the evolution, and the difference between linear and full Vlasov equation becomes negligible. In Landau theory, the avoidance of coherency and trapping is hence a key issue.

As shown in Refs. 10 and 15, a superposition of phase locked van Kampen modes all propagating with the same phase velocity v_0 and being composed of the same spectrum as the corresponding hole mode, satisfying hence the same macroscopic conditions as the hole, is formally possible, but it is not a solution of the full Vlasov equation and must be discarded.

A continuous transition between damped and growing coherent solutions demands a nonlinear equilibrium at transition, valid for the full Vlasov equation and bringing in the class of TPE and being characterized by a vanishing nonlinear growth rate γ_{NL}.

As an example, in case of immobile ions and kinetic electrons, the van Kampen mode has to be replaced by a CEH, see Refs. 10 and 15, to validate the continuous transition for the full Vlasov equation. The key point thereby is that a CEH remains distinct from a van Kampen mode for any amplitude, including the infinitesimal amplitude limit. The macroscopic identity of both solutions does not imply their microscopic identity. This is why linear wave theories as a whole inclusively Landau theory fail in predicting the onset of instability in case of a coherent wave pattern. The common assumption that linear wave theory has its legitimacy for sufficiently small amplitudes can simply not be made.

To substantiate these statements, let us consider a current driven plasma with a drift v_D between electrons and (mobile) ions. Linear wave theory then yields a critical line $v_D^*(\theta)$ in the $(\theta = \frac{T_e}{T_i}, v_D)$ parameter space describing the onset of instability. As seen from Figs. 2–4 of Ref. 37 and from Figs. 4–7 of Ref. 38 there exists below this line, i.e., for smaller v_D at given θ, a variety of hole solutions (solitary electron holes as well as solitary ion holes, harmonic waves, etc.) each of which being a potential candidate for destabilization, e.g., by a v_D somewhat larger than its own value. Since, however, the location of these hole structures depends also on quantities such as wave amplitude (ψ), electron (β), and ion (α) trapping parameters there is a whole band below $v_D^*(\theta)$ from which a plasma destabilization can start and take its origin. Hence, a whole band below critical drift velocity, $v_D < v_D^*$, can be a potential source of nonlinear instability, provided that it is triggered by an initial fluctuation in terms of a suitable seed-like depression in phase space or a non-topological fluctuation at resonant velocity (the latter being defined by its different slope against that of $f_0(v)$). The whole scenario gets further support by the fact that these holes are zero- or negative-energy holes[11,37,38] and are hence most easily excited. Special attention should thereby be paid to solitary ion holes, as for each θ an ion trapping parameter can be found for which virtually no minimum threshold v_D is needed for their existence, see Fig. 4 of Ref. 37. We finish by noting that there are meanwhile plenty of numerical simulations and analytical hints, which approve structure formation by nonlinear growth in subcritical two-stream plasma situations.[11,15,32–38]

VI. SUMMARY AND CONCLUSIONS

The subject of the present article have been undamped, 1D, weak, electrostatic structures, which typically propagate at bulk velocity in a thermal, collisionless plasma, i.e., in a region, where standard linear wave theory predicts non-existence due to strong Landau damping. The omnipresence of these structures in laboratory, space, and numerical experiments, however, indicates that something must be wrong with wave theories that rely on a linearization of the governing equations in the small amplitude limit and on a perturbative nonlinear analysis based on linear wave theory.

In the present paper, we have been shown by explicite construction and evaluation of coherent Vlasov–Poisson plasma equilibria how undamped structures in phase (and real) space can be brought to life by utilizing particle trapping. In this sense, coherency and linear wave theory do not go together. Particle trapping and the associated TN, responsible for non-analytic particle distribution functions, have been proved to be the driving force in the structure formation processes. Several singular physically impelled distributions have been considered, and corresponding conclusions have been drawn, e.g., the non-existence of solitary wave structures in cases of discontinuous distribution functions at the separatrix, including a void in the trapped region. As a rule, the simpler the wave structure is in its harmonic spectrum the stronger is the associated singularity. The known cnoidal electron (and ion) hole solutions are hence privileged as the ones coming closest to real physics providing the smoothest, albeit still non-analytic distribution functions, whereas BGK solutions typically exhibit a stronger singularity and are for this reason less relevant. It is worth mentioning that the analysis presented was straightforward and did not need any detour, such as a Fourier transformation and associated cut-offs, or restrictions, such as periodic boundary conditions, which may spoil the existence of solitary wave structures.

The arguments presented in this article hence ask for an additional path in wave theory supplementing Landau theory in which nonlinearity, stemming from trapping rather than from a hydrodynamic quadratic procedure (such as mode-coupling), prevails no matter how small the amplitude is. Trapping appears as an indispensable ingredient of proper descriptions of coherent structures and associated collisionless plasma dynamics which can be treated non-perturbatively only.

It is the velocity region at phase velocity—the resonant or trapped particle region—for which special care must be taken not to miss coherent structures. It is, in this respect, irrelevant how they have been generated, i.e., on which way from an initial seed the structure formation process has taken place in course of time, their long-time existence does allow such conclusions.

We finish with six remarks. First, the structures observed in a recent microinstability simulation,[39] in which the ruling role of the trapping nonlinearity was exposed, clearly give support to the present scenario. In it, the growth and saturation of a coherent electron hole could be seen propagating at ion acoustic velocity and being located at the rising wing of the drifting electron species in the current-driven plasma with $v_D > v_D^*$. A solitary ion acoustic wave and a solitary electron hole are in this regime one and the same object, namely, when electron trapping is less pronounced in comparison with the Boltzmannian state such that the dynamics is governed by the trapping nonlinearity. Second, the different opinions about the origin of structure formation, expressed recently in a dispute,[40,41] are resolved in favor of trapping, if there had been the need for yet another proof. Third, besides the observations in space, the most sensitive experimental measurements of collisionless phase space structures have probably been made in coasting and bunched beam experiments in storage rings such as in

Brookhaven, Fermilab, or Cern,[42,43] where long-living coherent structures could be detected, e.g., during rf-activity, circulating many times around the ring.[11,44] Fourth, quasi-stationary structures of this kind have been found in simulations of DC-driven weakly collisional plasmas with plateau-like trapped particle distributions as well, provided that ion mobility was taken into account.[9,11,35,45,46] They are hence representative of new dissipative out-of-equilibrium states of driven plasmas far away from thermodynamic equilibrium. Fifth, we point out that the nonlinear Landau damping scenario, mentioned in the Introduction, experiences a different fate when a non-perturbative structure formation process is admitted by the procedure as seen by the long term generation of tiny phase space vortices in the numerical simulations of Refs. 47, 11, and 35. Sixth, coarse-grained distribution functions in which singularities do no longer appear may be obtained by averaging over small cells in μ-space. Another approach may be achieved by invoking statistical mechanics principles such as Lynden-Bell's maximum entropy principle.[48] A successful test of the latter against a discrete simulation has been reported recently in Ref. 49 for the beam-plasma instability and the observed quasi-stationary-out-of-equilibrium states.

In conclusion, it is expected that the trapping scenario we have explored in some detail in the present paper by a non-perturbative analysis for electrostatic waves may give a clue for a deeper understanding of structural plasma turbulence in general and may thus contribute to the resolution of a long-standing mystery about their dynamical evolution.

APPENDIX A: DISCONTINUITY WITH OVERPOPULATED TRAPPED ELECTRONS

In Appendix A, we treat a discontinuous distribution with a surplus of trapped electrons at the separatrix $(\kappa > 1)$ and get from (8), (9) with (6) for $k_0^2 = 0$ and $b(v_0, \beta\kappa) = 0$

$$-\mathcal{V}(\Phi) = S\Phi^{3/2}(\sqrt{\Psi} - \sqrt{\Phi}) \tag{A1}$$

and

$$\frac{\kappa\gamma\sqrt{\pi}}{2} e^{-v_0^2/2} - \frac{1}{2} Z_r'\left(v_0/\sqrt{2}\right) = -2S, \tag{A2}$$

with

$$S := \frac{4(\kappa - 1)}{3\sqrt{\pi}\Psi} e^{-v_0^2/2} > 0. \tag{A3}$$

Ignoring the higher singularity in the trapped electron distribution, which has an influence only on v_0 but not on the shape, we set $\gamma = 0$ and get for the NDR

$$-\frac{1}{2} Z_r'\left(v_0/\sqrt{2}\right) = -2S < 0. \tag{A4}$$

From Fig. 1, we immediately see that only phase velocities above 1.307 are admitted and that it must hold $-2S > -0.285$ or $0 < S < 0.143$. A S in this interval yields two branches, a slow one with $1.307 < v_{0s} < 2.13$ and a fast one with $2.13 < v_{0f} < \infty$. Having obtained v_0 from the NDR, we can use the definition of S in (A3) to relate κ with Ψ, noting that $\Psi \ll 1$. We therefore get a κ close to unity. Only a small overpopulation is possible for a discontinuous distribution function and it holds $\kappa \to 1$ when $\Psi \to 0$.

To get the shape, we have to solve the integral

$$\int_\Phi^\Psi \frac{d\phi}{\sqrt{\phi^{3/2}(\sqrt{\Psi} - \sqrt{\phi})}} = \sqrt{2S}x, \tag{A5}$$

which follows from (4) with $\mathcal{V}(\Phi)$ given by (A1). This integral can be solved by the substitution of $u = \sqrt{\phi/\Psi}$ and a subsequent u-integration to yield

$$\Phi(x) = \Psi \cos^4\left(\frac{\sqrt{S}}{2\sqrt{2}}x\right), \tag{A6}$$

which is a periodic structure exhibiting narrow humps localized at the potential maxima. Although we assumed $k_0^2 = 0$, we did not get a pure solitary wave. Since, however, the trapping conditions can vary from one period to the next period, we can think of constructing a wavelet solution with a central peak and diminishing neighboring peaks (see also S12), all humps propagating with the same v_0. In the limit of only one hump, we then arrive at a solitary-like structure with a finite extension in x (solitary kind of wave having a "compact support").

We conclude that discontinuous distributions are in conflict with pure solitary waves and only admit periodic or wavelet structures.

APPENDIX B: ION ACOUSTIC SOLITON AS TRAPPED PARTICLE STRUCTURE

In Appendix B, it is shown that an ion acoustic soliton, satisfying a KdV equation, belongs to the class of trapped particle structures as well.

Its dynamics can first of all be formulated by using a cold ion fluid model since its phase velocity, the ion sound speed, by far exceeds the ion thermal velocity in case of hot electrons, the usual requirement for the existence of these structures. Trapping is instead found in the electron species the density of which being given by a Boltzmann relation in the isothermal limit. It becomes for small amplitudes

$$n_e = 1 + \phi + \phi^2/2 + \ldots. \tag{B1}$$

An extended expression, as derived in Refs. 3 and 28, allowing a generalization of the electronic trapping state, becomes (see Eq. (46) of Ref. 3 or Eq. (5) of Ref. 28)

$$n_e = 1 + \phi - \frac{4(1-\beta)}{3\sqrt{\pi}}\phi^{3/2} + \phi^2/2 + \ldots, \tag{B2}$$

which for isothermal electrons, $\beta = 1$, coincides with (B1). Note that the first three terms in (B2) are obtained from (5) for $\kappa = 1, k_0 = 0, v_0 = \sqrt{m_e/m_i} \approx 0$. Nonlinearity is introduced by the last term in both equations through the ordinary quadratic nonlinearity, whereas in (B2), an additional nonlinearity appears, the so-called TN, which is of $O(\phi^{3/2})$ and dominates the evolution unless β is close to unity.

For $\beta = 1$, the well known KdV-equation can then be derived, e.g., by means of a reductive perturbation method. For $\beta \simeq 0$, on the other hand, corresponding to a flat-topped electron distribution function, the corresponding evolution equation is of Schamel form (Ref. 28) given by

$$\partial_t \phi + \frac{(1-\beta)}{\sqrt{\pi}} \sqrt{\phi}\, \partial_x \phi + \frac{1}{2} \partial_x^3 \phi = 0, \tag{B3}$$

(see (14) of Ref. 28).

If $(1-\beta) = O(\sqrt{\psi})$, then both nonlinearities are of the same order and have to be treated on equal footing, yielding a nonintegrable SKdV equation (Refs. 3 and 28)

$$\partial_t \phi + \left[\frac{(1-\beta)}{\sqrt{\pi}} \sqrt{\phi} + \phi\right] \partial_x \phi + \frac{1}{2} \partial_x^3 \phi = 0. \tag{B4}$$

A solitary wave structure of (B4) is found in (48) of Ref. 3, turning into a sech^2-soliton in case of $\beta = 1$ and to a sech^4-solitary wave in case of $(1-\beta) = O(1)$. The ordinary ion acoustic soliton is hence a special case of a generalized solitary wave structure, the latter being distinguished by different trapped particle states. They are a generalization of the "maximum" trapped electron state, a state which is filled up "optimally" up to the isothermal Maxwellian state. We note in parenthesis that the finite amplitude expression corresponding to (B2) can be found in (2) of Ref. 28, turning into a Boltzmann expression for $\beta \to 1$. For further information, see also Ref. 10, especially Sec. IV A.

And last but not least, if it holds $(1-\beta) = O(\psi^{-1/2})$, corresponding to a still deeper excavation of the electron distribution, then both the linear and the nonlinear term are of equal size and contribute at the same level. Linear wave theory has lost its dominance. It is easily seen (see Sec. 4 of Ref. 3) that the corresponding solitary wave solution reads

$$\phi(x) = \psi\, \mathrm{sech}^4\left(\frac{x}{\Delta}\right) \quad \text{and} \quad v_0 = \frac{1}{\sqrt{1 - 16/\Delta^2}},$$
$$\text{where } \Delta := \sqrt{\frac{15\sqrt{\pi}}{(1-\beta)\sqrt{\psi}}} > 4.$$

Although $\psi \ll 1$, the TN term successfully competes with the linear term giving rise to a nonlinear wave solution, which has a small width of $O(1)$ independent of ψ and a larger phase speed, which has nothing to do anymore with the linear sound speed $v_0 = 1$. An example: $\Delta = 5$ yields $v_0 = 5/3$. A width of $O(1)$ of course undermines the applicability of a reductive perturbation method, which needs a lead of linearity and a stretched spatial extension.

So, already in the ion acoustic wave case, one can recognize the limitation and breakdown of standard wave theory and see how different trapping scenarios affect the wave solutions introducing a new wave model which has nothing in common anymore with the standard linearly rooted wave concept. Notice further that such a failure cannot be cured by letting the perturbation amplitude go to infinitesimal values as falsely believed in the common wave literature. Hence, all wave theories, which are based in their implementation on a Boltzmannian electron response in 1D, i.e., in which implicitly use is made of the existence of such an inherent "maximum" trapped electron state, can experience such a transformation (shift) away from standard wave approach by adopting different trapping states. Physically, as seen later, this appears to be especially important with respect to the ion species.

Finally, we mention the close relationship between nonlinear ion acoustic waves and electron holes referring to the first remark in Sec. VI and stress the symbiotic, contrarotating relationship between the microscopic trapped particle population (TPP) and the macroscopic TPN. A zero TPP implies a maximum TPN and a "maximum" TPP a zero TPN, the latter representing the door to the common wave approaches.

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