SUPPLEMENT: A Second Independent Proof Of The Magnetosonic Wave Collpase

Abstract: In this supplement the correctness of the results of [PRL] is independently proven by means of a simplified description that utilizes the Lagrangian mass variable.

The system of coupled differential eqs. (5),(6) of the PRL article

[PRL] N. Chakrabarti, Ch. Maity, and H. Schamel, Phys. Rev. Lett. 106, 145003 (2011)

can be substantially simplified- without loss of generality - by switching to the Lagrangian mass variable, denoted by η. Taking a look at
- H.Schamel, Phys.Reports 392(2004)279 - one can see that through (15) of this paper the introduction of η instead of ξ yields to the differential operator
$\frac{\partial}{\partial \eta} = \frac{1}{n(\xi,0)} \frac{\partial}{\partial \xi}$.
Moreover, if one introduces for convenience the specific volume $V(\eta,\tau) = \frac{1}{n(\eta,\tau)}$, which is by the way the Jacobian of the transformation from (x,t) to (η,τ), one finds that (5),(6) in [PRL] become

$V_{\tau\tau} + (B^2/2)_{\eta\eta} = 0$ (5'),

$(BV)_\tau = (\epsilon B_\tau + \sigma B)_{\eta\eta}$ (6')

where in (6') use has been made of the symbol σ for the resistivity replacing the earlier dissipation parameter η.
One realizes that the space dependent coefficient $n(\xi,0)$ has now disappeared , leaving the system in a substantially simpler form.
The expectation is that through this simpler system a more straightforward solution can be obtained, recovering hopefully the earlier solution.

A solution of (5'),(6') can be attempted by a separation ansatz.
Set $V(\eta,\tau) := a(\tau)\mathcal{V}(\eta)$ and $B(\eta,\tau) := b(\tau)\mathcal{B}(\eta)$
to get from (5'),(6')

$\frac{\ddot{a}}{b^2} = -\alpha = -\frac{(\mathcal{B}^2/2)''}{\mathcal{V}}$ (7')
$\frac{a\dot{b}+\dot{a}b}{(\epsilon\dot{b}+\sigma b)} = \beta = \frac{\mathcal{B}''}{\mathcal{V}\mathcal{B}}$ (8')

With this one is in accordance with the PRL article, where $\alpha = 2$ and $\beta = \pi/2$ had been chosen to get the solution.

In a first step one can try to recover the PRL solution (9)-(11) for the spatial and temporal part by setting $\alpha = 2$ and keeping β.(Later one can can set $\beta = \pi/2$.) The hope is that one can confirm the previous solution and learn how to proceed to get new ones. Of special interest is to learn which role different constants α and β do play.

Concentrate first on the temporal part because it is this part, which should not be affected by the space transformation

$\ddot{a} + 2b^2 = 0 \qquad$ (A)

$\dot{(ab)} = \dot{a}b + a\dot{b} = \beta(\epsilon\dot{b} + \sigma b)\ . \qquad$ (B)

(i) first, the ideal non-resistive part is treated by setting $\sigma = 0$.
A solution of (B) can immediately be found:
$b = \frac{1-\beta\epsilon}{a-\beta\epsilon}\ , \qquad$ (C)
where the initial condition $a(0) = 1 = b(0)$ was used. Substitution into (A) and integration (by multiplying (A) by $\dot{a}$ and integration) yields
$\dot{a} = -\frac{2(1-\beta\epsilon)}{\sqrt{a-\beta\epsilon}},$
where the negative branch of the square root was selected and $a \geq \beta\epsilon$ was assumed. A further integration gives
$a(\tau) = \beta\epsilon + \left[(1-\beta\epsilon)^{3/2} - 3c_3\tau\right]^{2/3} \qquad$ (D),
where $a(0) = 1$ was used and $c_3 = (1-\beta\epsilon)$.
(D) shows that $a(\tau)$ is decreasing with increasing time, reaching $a(\tau_c) = \beta\epsilon$ with infinite slope ($\dot{a} \to \infty$) , as τ approaches $\tau_c := \frac{(1-\beta\epsilon)^{3/2}}{3c_3}$.
Since a stands for the specific volume and $1/a$ for the density one sees that the density is compressed (i.e. increases) in time but stays finite as long as ϵ, the dispersion, is finite. If, on the other hand, the dispersion disappears, i.e. $\epsilon \to 0$, a becomes zero and the density diverges: the density collapses at τ_c, the collapse time.
Going back to the magnetic field (C) one sees that b becomes ∞ in either case in this limit. The magnetic field collapses independent of dispersion. Dispersion is able to stop the density but not the magnetic field collapse.
With this one has reproduced the ideal PRL results with respect to the time behavior.

The question now arises as to whether the results can also be confirmed for finite resistivity, where a delaying effect was predicted .

(ii) second, finite resistivity is treated by letting $\sigma \neq 0$.
The difficulty can immediately be seen through (B), where finite σ destroys the direct integrability of (B). A way around this dilemma was proposed in the PRL article through the integrating factor, introduced by
$\frac{1}{b(\tau)}\frac{d}{d\tau} = f(\theta)\frac{d}{d\theta}$ (E)

If $f(\theta)$ is assumed to be exponential (in accordance with [PRL])
$f(\theta) = \exp(A\theta)$ (F)
(B) can be integrated, yielding
$\hat{b}(\theta) = \frac{1}{(\hat{a}(\theta)-\beta\epsilon)}[(1-\beta\epsilon) + \frac{\beta\sigma}{A}(1-\exp(-A\theta))]$ (G)
Notice that in case of $\sigma = 0$ the old result (C) is recovered with $c_3 = (1-\beta\epsilon)$.
The solution presented in PRL (noting that $a = 1/\phi$ and $b = \psi$) reads:

$$\hat{b}(\theta) = sech^2(\frac{\theta}{\sqrt{\beta\epsilon-1}})\exp(\frac{\sigma\beta}{\beta\epsilon-1}\theta) \qquad \text{(PRL1)}$$

$$\hat{a}(\theta) - \beta\epsilon = (1-\beta\epsilon)cosh^2(\frac{\theta}{\sqrt{\beta\epsilon-1}})\exp(\frac{2\sigma\beta}{1-\beta\epsilon}\theta) \qquad \text{(PRL2)}$$

It is easily seen that these two expressions satisfy (G) when
$A = \frac{\beta\sigma}{\beta\epsilon-1}$. (H)
This is a first hint that the PRL results are correct. What remains to be shown for the final proof is that also (A) is satisfied.

Switching from τ to θ by means of (E) and (F) one gets
$\frac{d}{d\tau} = sech^2(\frac{\theta}{\sqrt{\beta\epsilon-1}})\exp(2A\theta)\frac{d}{d\theta}$,
which, inserted into (A) and making use of (PRL1) for $\hat{b}$, yields

$$\hat{a}'' + (2A - \frac{2}{\sqrt{\beta\epsilon-1}}tanh\frac{\theta}{\sqrt{\beta\epsilon-1}})\hat{a}' + 2\exp(-2A\theta) = 0.$$

A straightforward calculation then shows that this equation is indeed satisfied by (PRL2) with A given by (H). This confirms the correctness of the results in the PRL article with respect to the temporal behavior, which is the essential part because it decides about its collapse behavior.

Sept. 18, 2012 H.S.